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Some Solvable Eigenvalue Problems

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An extension of the concept of supersymmetrization is proposed in which the couple of separated second-order differential equations can be fit into a new scheme with existence of a double degeneracy of their eigenspectra. As illustration of the method, some exactly solvable problems related to the $U(1, 1)$ group are discussed explicitly.

Conventional supersymmetric systems are usually considered as part of the classes of either unbroken or broken symmetry. The first category is related to the existence of a positive-energy ground state leading to a strictly complete degeneracy for the eigenspectrum, for instance, in the $SU(2)$ case. The second one means existence of a zero-energy ground state which is not degenerate so that the double degeneracy is not complete. If Δ is the Witten (1991) index, then $\Delta = 0$ for the first case and $\Delta = \pm 1$ for the second one.

For clarity, we recall first some definitions and notations about conventional supersymmetry. Consider two generators A_1, A_2 and two matrices Q_1, Q_2:

$$A_1 = \frac{d}{dx} + v, \qquad A_2 = \frac{d}{dx} - v$$

$$Q_1 = \begin{pmatrix} 0 & 0 \\ A_1 & 0 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 0 & A_2 \\ 0 & 0 \end{pmatrix}$$

and consider the system (ψ_1, ψ_2) such that

$$A_1\psi_1 = k\psi_2, \qquad A_2\psi_2 = k\psi_1 \tag{1}$$

k is a constant parameter, and v is a function usually referred to as the "superpotential." These quantities obey the "graded algebra"

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$$Q_1^2 = Q_2^2 = 0 \tag{2}$$

$$\{Q_1, Q_2\} = H, \qquad H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \tag{3}$$

$$H_1 = A_2 A_1, \qquad H_2 = A_1 A_2$$

$$[Q_1, H] = [Q_2, H] = 0 \tag{6}$$

The symbols { , } and [,] mean anticommutation and commutation operations, respectively. Expression (2) corresponds to nilpotency, while (4) denotes conservation of the "supercharges" Q_1 and Q_2; H is the Hamiltonian of the system. The quantity v can be defined as

$$\frac{1}{2}\{A_2, A_1\} = \frac{d^2}{dx^2} - v^2$$

$$\frac{1}{2}[A_2, A_1] = -v' \tag{5}$$

Consider now another system $(\overline{\psi}_1, \overline{\psi}_2)$ and the generators $\overline{A}_1, \overline{A}_2$ defined as

$$\overline{A}_1 = (f_1 f_2)^{1/2}\left(\frac{d}{dx} + \left(u - \frac{1}{2}\frac{f_1'}{f_1}\right)\right), \qquad \overline{A}_2 = (f_1 f_2)^{1/2}\left(\frac{d}{dx} - \left(u + \frac{1}{2}\frac{f_2'}{f_2}\right)\right)$$

u, f_1, f_2 may in principle be any analytic functions. Let

$$\overline{Q}_1 = \begin{pmatrix} 0 & 0 \\ \overline{A}_1 & 0 \end{pmatrix}, \qquad \overline{Q}_2 = \begin{pmatrix} 0 & \overline{A}_2 \\ 0 & 0 \end{pmatrix}$$

Then

$$\overline{Q}_1^2 = \overline{Q}_2^2 = 0 \tag{2b}$$

$$\{\overline{Q}_1, \overline{Q}_2\} = \overline{H}, \qquad \overline{H} = \begin{pmatrix} \overline{H}_1 & 0 \\ 0 & \overline{H}_2 \end{pmatrix} \tag{3b}$$

$$\overline{H}_1 = \overline{A}_2 \overline{A}_1, \qquad \overline{H}_2 = \overline{A}_1 \overline{A}_2$$

$$[\overline{Q}_1, \overline{H}] = [\overline{Q}_2, \overline{H}] = 0 \tag{4b}$$

Let

$$\overline{A}_1 \overline{\psi}_1 = k\overline{\psi}_2, \qquad \overline{A}_2 \overline{\psi}_2 = k\overline{\psi}_1$$

Then

$$\overline{H}_1 \overline{\psi}_1 = k^2 \overline{\psi}_1, \qquad \overline{H}_2 \overline{\psi}_2 = k^2 \overline{\psi}_2$$

with

$$\overline{H}_1 = f_1 f_2 \left\{ \frac{d^2}{dx^2} - \left[\left(u - \frac{1}{2}\frac{f_1'}{f_1} \right)^2 - \left(u' - \frac{1}{2}\left(\frac{f_1'}{f_1}\right)' \right) \right] \right\}$$

$$\overline{H}_2 = f_1 f_2 \left\{ \frac{d^2}{dx^2} - \left[\left(u + \frac{1}{2}\frac{f_2'}{f_1} \right)^2 + \left(u' + \frac{1}{2}\left(\frac{f_2'}{f_2}\right)' \right] \right\} \tag{6}$$

It is instructive to point out that the result (6) was derived from the generalized theory of coupled differential equations (Cao, 1992, 1994) and the present development can be regarded as consequences of this theory.

We have here the "graded algebra," which, however, is not exactly identical to Witten's case, since the commutation relations equivalent to (5) cannot be expressed in a simple manner. We find

$$\frac{1}{2}\{\overline{A}_2, \overline{A}_1\} = f_1 f_2 \left[\frac{d^2}{dx^2} - u^2 - \frac{1}{8}\left(\left(\frac{f_1'}{f_1}\right)^2 + \left(\frac{f_2'}{f_2}\right)^2 \right) \right.$$

$$\left. - \frac{1}{2} u \left(\frac{f_2'}{f_2} - \frac{f_1'}{f_1} \right) - \frac{1}{4}\left(\left(\frac{f_1'}{f_1}\right)' + \left(\frac{f_2'}{f_2}\right)' \right) \right] \tag{5b}$$

$$\frac{1}{2}[\overline{A}_2, \overline{A}_1] = f_1 f_2 \left[\frac{1}{8}\left(\left(\frac{f_1'}{f_1}\right)^2 - \left(\frac{f_2'}{f_2}\right)^2 \right) \right.$$

$$\left. + \frac{1}{2} u \left(\frac{f_1'}{f_1} + \frac{f_2'}{f_2} \right) + u' + -\frac{1}{4}\left(\left(\frac{f_2'}{f_2}\right)' - \left(\frac{f_1'}{f_1}\right)' \right) \right]$$

These results enable us to extract two significant remarks concerning the choice of the functions f_1, f_2.

1. f_1 and f_2 are assumed to be constants.
2. The are inverse functions in the sense $f = f_1 = \alpha/f_2$, α being a constant.

For remark 1 we can see that (5b) become identical to (5) if $u = v$ or in other words, u can be identified as the superpotential. For remark 2, in which $f_1'/f_1 = -f_2'/f_2$, this superpotential is $v = u - \frac{1}{2}f'/f$. These cases do not bring anything new because they are merely different aspects of the Witten formulation.

Generalization

Obviously, when the functions f_1 and f_2 are not governed by the constraints 1 or 2, the usual concept of superpotential become invalid and the situation requires a new approach.

Here we focus on some simple situations where the system can be solved exactly.

The simplest situation concerns the choice $f_1 = f_2 = f$ with $u = a_1 + Bf'/f$, a_1 and B being constant parameters.

The Hamiltonians $\overline{H}_1$ and $\overline{H}_2$ defined in (6) become

$$\overline{H}_1 = f^2\left[\frac{d^2}{dx^2} - V_1(a_1, B, x)\right], \qquad \overline{H}_2 = f^2\left[\frac{d^2}{dx^2} - V_2(a_1, B, x)\right]$$

$$f^2V_1(A_1, B, x) = A_1^2 f^2 + \left(B^2 - \frac{1}{4}\right)f'^2 + 2A_1\left(B - \frac{1}{2}\right)f'f - \left(B - \frac{1}{2}\right)f''f$$

$$f^2V_2(A_1, B, x) = A_1^2 f^2 + \left(B^2 - \frac{1}{4}\right)f'^2 + 2A_1\left(B + \frac{1}{2}\right)f'f + \left(B + \frac{1}{2}\right)f''f \tag{7}$$

Consider two types of constraints:

$$\begin{aligned}&\text{(a)}\quad f'^2 - f''f = +1\\ &\text{(b)}\quad f'^2 - f''f = -1\end{aligned} \tag{8}$$

leading to five cases:

	(a-1)	(a-2)	(a-3-1)	(a-3-2)	(b-1)
$f(x)$	x	sh x	sin x	cos x	ch x

Note the equivalence of (a-3-1) and (a-3-2) relative to the shift $x \to x \pm \pi/2$.

They lead to a family of five Hamiltonians, which in fact, may have some similarities, which will be examined through a single unified approach. From (8) it is found that ($f'^2 = t + lf^2$)

$$f^2V_1(a_1, B, x) = \left[a_1^2 + l\left(B - \frac{1}{2}\right)^2\right]f^2 + 2a_1\left(B - \frac{1}{2}\right)f'f + t\left(B^2 - \frac{1}{4}\right) \tag{9}$$

$$f^2V_2(a_1, B, x) = \left[a_1^2 + l\left(B + \frac{1}{2}\right)^2\right]f^2 + 2a_1\left(B + \frac{1}{2}\right)f'f + t\left(B^2 - \frac{1}{4}\right)$$

where two parameters l and t have been included and are defined as follows:

	(a-1)	(a-2)	(a-3-1)	(a-3-2)	(b-1)
l	0	+1	−1	+1	+1
t	+1	+1	+1	+1	−1

Exact Solutions

The eigenfunctions of (6) can be expanded in the form

$$\overline{\psi}_j^{(i)} = \sum_{m=1} a_{j,m}^{(i)} \left\{ \exp\left[\frac{a_1\left(B \mp \frac{1}{2}\right)}{r_j^{(i)} + m} x \right] \right\} f^{r_j^{(i)}+m} \tag{10}$$

with the following notations: the $\mp$ signs refer to $j = 1, 2$; (i) denotes the five cases (a-1), (a-2), . . . ; $r_j^{(i)}$, m are parameters.

From now on, these indices will be omitted for simplicity.

Replacing (10) in (6) and using (8), it can be verified that the coefficients $a_{j,m}^{(i)}$ must obey a two-term recursion relation

$$l^{\frac{a_1\left(B - \frac{1}{2}\right)}{(r + m - 2)} x} a_{m-2} \left[l(r + m - 2)^2 - \left(a_1^2 + l\left(B - \frac{1}{2}\right)^2\right) + \frac{a_1^2\left(B - \frac{1}{2}\right)^2}{(r + m - 2)^2} \right]$$

$$+ e^{\frac{a_1\left(B - \frac{1}{2}\right)}{r + m} x} a_m \left[t\left((r + m)(r + m - 1) - \left(B^2 - \frac{1}{4}\right)\right) - k^2 \right] = 0 \tag{11}$$

which can be solved by standard methods:

For the case $m = 0$, let $a_0 \neq 0$ and $a_1 = 0$. The eigenvalues are given by

$$k_N^2 = t\left[r(r - 1) + \frac{1}{4} - B^2 \right] \tag{12}$$

If $m = N + 2$, let $a_N \neq 0$ and $a_{N+2} = \cdots = 0$; then the parameter r must be a solution of the equation

$$l(r + N)^2 - \left(a_1^2 + l\left(B \mp \frac{1}{2}\right)^2\right) + \frac{a_1^2\left(B \mp \frac{1}{2}\right)^2}{(r + N)^2} = 0 \tag{13}$$

Two kind of solutions are possible:

$$(r + N)^2 = \begin{cases} (1/l)a_1^2 \\ \left(B \mp \frac{1}{2}\right)^2 \end{cases}$$

The first one is dependent on the parameter l, while the second one is independent of both l and t.

Discussion

For the cases (a-3-1) and (a-3-2) (the trigonometric functions) l is negative, so that the first kind must be excluded. One is left with only one possible solution,

$$r = \left(B \mp \frac{1}{2}\right) - N \tag{14}$$

This result is also valid for the case (a-1), where $l = 0$.

The hyperbolic cases (a-2), (b-1) with l positive lead to two solutions:

$$r = \begin{cases} a_1 - N \\ \left(B \mp \frac{1}{2}\right) - N \end{cases} \tag{15}$$

The Spectrum

I. For the cases (a-1), (a-3-1), (a-3-2) with (14), we obtain

$$k_{1,N}^2 = 2(N + 1)\left[\frac{N + 1}{2} - B\right], \qquad k_{2,N}^2 = 2N\left[\frac{N}{2} - B\right] \tag{16}$$

In other words,

$$k_{2,N}^2 = k_{1,N-1}^2$$

which expresses the double degeneracy except for the ground state ($N = 0$). This is the case of unbroken symmetry discussed above and denoted by ($k_{u,N}^2$).

II. The case (a-2) has two possible eigenspectra, so that in addition to the result (16) there is a second one given by

$$k_{1,N}^2 \equiv k_{2,N}^2 = -B^2 + \frac{1}{4} + (a_1 - N)(a_1 - N - 1) \tag{17}$$

which indicates a complete degeneracy related to broken symmetry ($k_{b,N}^2$).

III. Finally, for the case (b-1), we have the same type of eigenspectrum (17), but the spectrum of type (16) is ≈

$$k_{1,N}^2 = -2N\left(\frac{N}{2} - B\right), \qquad k_{2,N}^2 = -2(N - 1)\left(\frac{N - 1}{2} - B\right)$$

Shape Invariance

Returning to (9), it can be seen that

$$f^2V_1(a_1, B, x) = f^2V_1(a_1, B - 1, x) + t((B - 1)^2 - B^2) \tag{18}$$

which expresses the shape invariance condition of Gedenshtein (1983), so the eigenspectra can be inferred alternatively from the relation

$$k_N^2 = t \cdot 2N\left[\frac{N}{2} - B\right] \tag{19}$$

The parameter t is given in the table above. As expected, these results are in exact agreement with the ones found by a direct approach for the spectra of type $(k_{u,N}^2)$.

We may relate the present formulation to the model suggested earlier by Lahiri *et al.* (1988), who imposed a four-parameter group structure in the ladder operators A, A^+

$$A = e^{iy}\left[f(x)\frac{\partial}{\partial x} - if'(x)\frac{\partial}{\partial y} + v(x)\right]$$

$$A^+ = e^{-iy}\left[-f(x)\frac{\partial}{\partial x} - if'(x)\frac{\partial}{\partial y} + v(x)\right]$$

$v(x), f(x)$ are arbitrary functions and y is considered as an auxiliary parameter. These operators obey the following algebra:

$$A_3 = -i\frac{\partial}{\partial y}, \qquad [A, A^+] = -2a\, A_3 - bI$$

$$[A_3, A] = A, \qquad [A_3, A^+] = -A^+$$

with

$$a = f'^2 - f''f, \qquad b = 2(f'v - fv')$$

The Hamiltonians are defined as $H = \frac{1}{2}\{A, A^+\}$ corresponding to the eigenvalue equation

$$Hg(x, y) = k^2 g(x, y)$$

where $g(x, y)$ is separable in the sense

$$g(x, y) = \psi(x)e^{iny}$$

n is a parameter.

For the special case where $a = 1$ and $b = 0$, A, A^+ can be identified with the generators of a $U(1, 1)$ group.

Three types of Hamiltonians were considered by Janussis *et al.* (1990) and Samantha (1993) with $f = x$, $\sin \mu x$, $\cos \nu x$, with μ and ν constant parameters; a complete solution of the first case was given.

The following remarks may then be instructive:

I. In the present work, it can be verified that with an appropriate choice of the quantities a_1, B the function $\psi(a)$ becomes in fact identical to the eigenfunctions $\bar{\psi}$.

More precisely, one must take $u(x) = a_1 f(x)$, f being one of the five functions mentioned above. For instance, for $f^2 V_1$, in the case (a-3-1), the choice will be $a_1 = 1$, $B = n + \frac{1}{2}$.

II. The interest of the present formulation is therefore:

(a) The case (a-2) can be added to the list of functions mentioned in Janussis *et al.* (1990) and Samantha (1993).

(b) The case (a-1), which had already been solved by these authors, is confirmed in the present work.

(c) These five cases can be approached in the frame of a single unified method which leads to a similar two-term recursion relation.

(d) The concept of shape invariance remains valid.

(e) The broken and unbroken symmetry is conventional supersymmetrization are mainly related to the parity of the superpotential (Cao, 1990; Ralchenko and Semenov, 1992; Dutt *et al.*, 1993), while the present approach obviously is not subject to this constraint. This may justify in a sense the occurrence of the two types of eigenspectrum in cases (a-2) and (b-1).

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